Section 11-3

The Dot Product of Two Vectors

Definition of Dot Product:

$\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

$=$ scalar.

$\Rightarrow$ In a similar fashion:

if $\vec{u} = \langle u_1, u_2, u_3 \rangle$

and $\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Class B

1) DA Late

2) David Late

Properties of Dot Products

1) Commutative Property

$$\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$$

2) Distributive Property

$$\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$$

3) $c(\vec{u}\cdot\vec{v})=c\vec{u}\cdot\vec{v}=\vec{u}\cdot c\vec{v}$

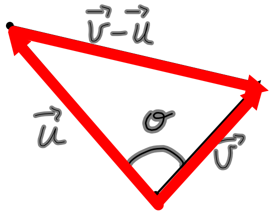
4) $\vec{0}\cdot\vec{v}=0$

5) $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2$

Angle Between Two Vectors

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$$

Proof: Simply use Law of Cosines.



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\left(\|\vec{v}-\vec{u}\|\right)^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

Since $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

$$\Rightarrow \|\vec{v}-\vec{u}\|^2 = (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})$$
$$= \vec{v}\cdot\vec{v} - 2\vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{u}$$
$$= \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{u}\|^2$$

$$\therefore \cancel{\|\vec{v}\|^2} - 2\vec{u}\cdot\vec{v} + \cancel{\|\vec{u}\|^2} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

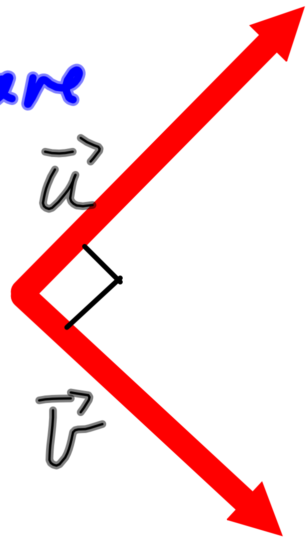
$$-2\vec{u}\cdot\vec{v} = -2\|\vec{u}\|\|\vec{v}\|\cos\theta$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

Definition of Orthogonal Vectors

The vectors u and v are orthogonal if

$$\vec{u} \cdot \vec{v} = 0$$



*NOTE: Orthogonal is same as perpendicular

Note:

Orthogonal - used for vectors

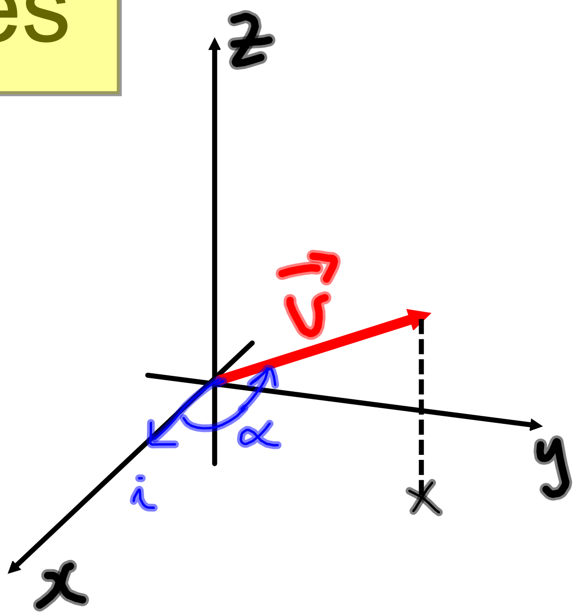
Perpendicular - used for lines

Normal - used for vector and line or vector and plane.

All essentially mean "at right angles"

Direction Cosines

$\cos\alpha = \dfrac{\vec{v}_1}{\|\vec{v}\|}$
$\cos\beta = \dfrac{\vec{v}_2}{\|\vec{v}\|}$
$\cos\gamma = \dfrac{\vec{v}_3}{\|\vec{v}\|}$



* Where:

α is the angle between i and $\vec{v}$,

β is the angle between j and $\vec{v}$,

γ is the angle between k and $\vec{v}$,

$$v \cdot i = \|v\| \|i\| \cos\alpha$$

$$v \cdot i = \langle v_1, v_2, v_3 \rangle \cdot \langle 1, 0, 0 \rangle$$

$$= v_1$$

$$\cos\alpha = \frac{v_1}{\|v\|}$$

Definitions of Projection and Vector Components

1) $\boxed{\vec{w}_1 = \text{proj}_{\vec{v}}\,\vec{u}}$

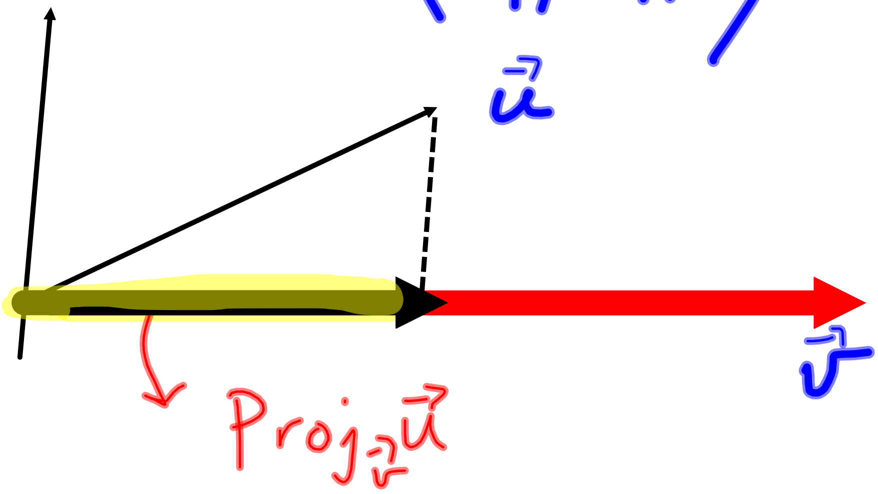
= The projection of $\vec{u}$ onto $\vec{v}$.

2) $\boxed{\vec{w}_2 = \vec{u} - \vec{w}_1}$

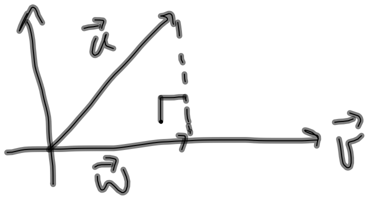
= Vector component of $\vec{u}$ orthogonal to $\vec{v}$.

Projection Using the Dot Product

$$\text{Proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$



Proof (for fun - you don't need to know this)



$$\cos\theta = \frac{\|\vec{w}\|}{\|\vec{u}\|} \begin{matrix} \leftarrow \text{adj} \\ \leftarrow \text{hyp} \end{matrix}$$

$$\|\vec{w}\| = \|\vec{u}\|\cos\theta \qquad \$$$

since

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

so $\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ ¥

multiply \$ by $\frac{\|\vec{v}\|}{\|\vec{v}\|}$

$$\|\vec{w}\| = \frac{\|\vec{u}\|\,\|\vec{v}\|\cos\theta}{\|\vec{v}\|}$$

From ¥

$$\|\vec{w}\| = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

since we want $\vec{w}$ to be parallel to $\vec{v}$:

$\|\vec{w}\|$

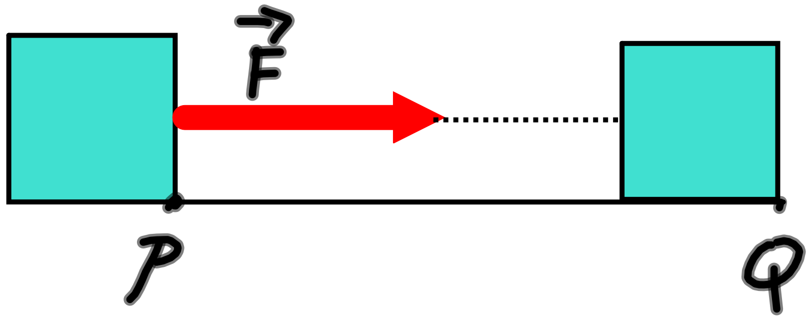
$$\vec{w} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

$$= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

Work

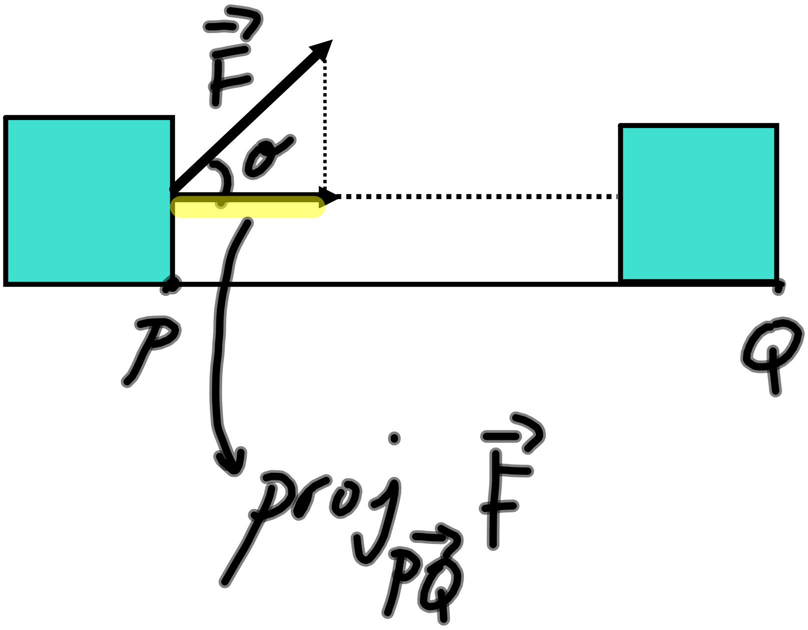
The work, W, done by the constant force F acting ALONG the line of motion of an object is:

$$\boxed{W = \vec{F} \cdot \vec{PQ}}$$



If the constant force is NOT directed along the line of motion, work must be calculated using:

$$\boxed{W = \left\| \text{proj}_{\vec{PQ}} \vec{F} \right\| \left\| \vec{PQ} \right\|}$$



Examples: Page 790

Find: a) u dot v b) u dot u, c) ||u||^2, d) (u dot v)v, e) u dot (2v)

3. $\mathbf{u} = \langle 6, -4 \rangle$, $\mathbf{v} = \langle -3, 2 \rangle$

a) $u \cdot v = -18 - 8 = -26$

b) $u \cdot u = 36 + 16 = 52$

c) $||u||^2 = \left(\sqrt{36+16}\right)^2 = 52$

d) $(u \cdot v)v = -26 \langle -3, 2 \rangle = \langle 78, -52 \rangle$

e) $u \cdot 2v = 2\,(u \cdot v) = 2\,(-26)$

$= -52$

Find: a) u dot v b) u dot u, c) ||u||^2, d) (u dot v)v, e) u dot (2v)

5. $\mathbf{u} = \langle 2, -3, 4 \rangle, \mathbf{v} = \langle 0, 6, 5 \rangle$

(a) $\mathbf{u} \cdot \mathbf{v} = 2(0) + (-3)(6) + (4)(5) = 2$

(b) $\mathbf{u} \cdot \mathbf{u} = 2(2) + (-3)(-3) + 4(4) = 29$

(c) $\|\mathbf{u}\|^2 = 2^2 + (-3)^2 + 4^2 = 29$

(d) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 2\langle 0, 6, 5 \rangle = \langle 0, 12, 10 \rangle$

(e) $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(2) = 4$

Extend Page

Find the angle theta between the vectors

13. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$

$u = \langle 3, 1, 0 \rangle$

$v = \langle -2, 4, 0 \rangle$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{-2}{\sqrt{10}\sqrt{20}}$$

$\sqrt{9+1}$ $\sqrt{4+16}$

$$\theta = \cos^{-1}\left(\frac{-2}{\sqrt{200}}\right) = \boxed{98.1^\circ}$$

Find the angle theta between the vectors

17. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$

$u = \langle 3, 4, 0 \rangle$

$v = \langle 0, -2, 3 \rangle$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$$

$\theta = 116.344$ ✓

Find a) the projection of u onto v, and b) find the vector component of u orthogonal to v.

43. $\mathbf{u} = \langle 6, 7 \rangle$, $\mathbf{v} = \langle 1, 4 \rangle$

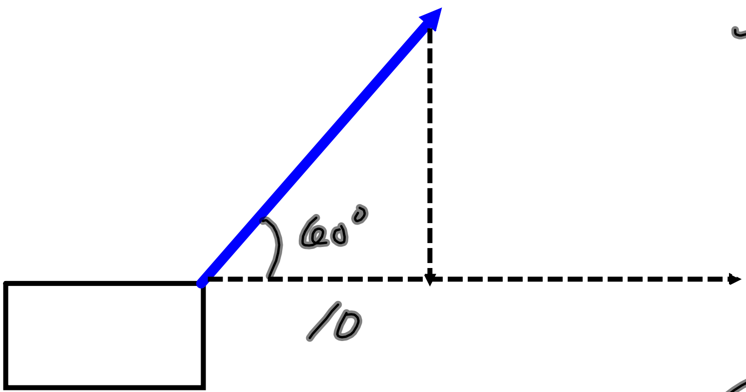
(a) $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

$$= \frac{6(1) + 7(4)}{1^2 + 4^2}\langle 1, 4 \rangle$$

$$= \frac{34}{17}\langle 1, 4 \rangle = \langle 2, 8 \rangle$$

(b) $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \langle 6, 7 \rangle - \langle 2, 8 \rangle = \langle 4, -1 \rangle$

73) WORK: An object is pulled 10 feet across a floor, using a force of 85 pounds. The direction of the force is 60 degrees above the horizontal.
Find the work done.



$x = \cos\theta$

$x = \cos 60^\circ$

$x = \frac{1}{2}$

$y = \sin\theta$

$y = \sin 60^\circ$

$= \frac{\sqrt{3}}{2}$

$$73.\quad \mathbf{F} = 85\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$$

$$\mathbf{v} = 10\mathbf{i}$$

$$W = \mathbf{F}\cdot\mathbf{v} = 425 \text{ ft-lb}$$

75) WORK. A car is towed using a force of 1600 newtons. The chain used to pull the car makes a 25 degree angle with the horizontal. Find the work done in towing the car 2 kilometers.

$$\mathbf{F} = 1600(\cos 25°\,\mathbf{i} + \sin 25°\,\mathbf{j})$$

$$\mathbf{v} = 2000\mathbf{i}$$

$$W = \mathbf{F} \cdot \mathbf{v} = 1600(2000)\cos\ 25°$$

$$\approx 2{,}900{,}184.9 \text{ Newton meters (Joules)}$$

$$\approx 2900.2 \quad \text{km-N}$$

HW#71 (11-3)

Pg 789# 2,12,14,18,20,24,26